Journal of Sound and Vibration 329 (2010) 1985–1991

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Rapid Communication

Pole assignment of friction-induced vibration for stabilisation through state-feedback control

Huajiang Ouyang *

Department of Engineering, University of Liverpool, The Quadrangle, Brownlow Street, Liverpool L69 3GH, UK

A R T I C L E   I N F O

Article history:
Received 24 September 2009
Received in revised form
13 December 2009
Accepted 14 December 2009
Handling Editor: A.V. Metrikine
Available online 13 January 2010

A B S T R A C T

Friction-induced self-excited linear vibration is often governed by a second-order matrix differential equation of motion with an *asymmetric* stiffness matrix. The asymmetric terms are product of friction coefficient and the normal stiffness at the contact interface. When the friction coefficient becomes high enough, the resultant vibration becomes unstable as frequencies of two conjugate pairs of complex eigenvalues (poles) coalesce (when viscous damping is low).

This short paper presents a receptance-based inverse method for assigning complex poles to second-order asymmetric systems through (active) state-feedback control of a combination of active stiffness, active damping and active mass, which is capable of assigning negative real parts to stabilise an unstable system.

1. Introduction

Friction dissipates heat energy and hence is a major damping mechanism. However, it is also capable of doing the opposite, that is, exciting and sustaining vibration [1,2]. For example, brakes can generate all sorts of noise as a result of friction-induced vibration [3]. Brake noise and other friction-related noise were reviewed by Akay [4]. Friction-induced vibration is very difficult to mitigate as it depends on and is sensitive to a number of factors [1,5,6].

Discretised linear or linearised models of friction-induced vibration are governed by a second-order matrix differential equation with an asymmetric stiffness matrix. Two typical models were discussed in Ref. [7]. For both models, when friction coefficient increases to a certain high enough value, the imaginary parts (frequencies) of two conjugate complex pairs of eigenvalues coalesce and the real part of one pair becomes positive [8]. As a result, the system becomes unstable (flutter instability occurs). Viscous damping in asymmetric systems is found to be mostly destabilising [9,10], contrary to the intuition. Therefore, viscous damping should be included in the governing equation of friction-induced vibration. Incidentally, von Wagner et al. [11] discussed several interesting ways of introducing friction into a dynamic model, including their own.

One passive means of vibration control is to assign desired eigenvalues through structural modifications [12]. They have been widely used for symmetric dynamic systems [13,14]. However, it is well known that passive vibration control has inherent limitations. These include undesirable changes of unassigned eigenvalues and the restriction that the rank of modifications must not be smaller than the number of assigned frequencies and zeros. Active control has been shown to be able to overcome these difficulties, as shown, for example, in Refs. [15–19]. Ram and Mottershead were the first

* Fax: +44 151 794 4848.
E-mail address: h.ouyang@liverpool.ac.uk

0022-460X/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2009.12.027

researchers to introduce a receptance-based inverse method to assign poles and zeros to *symmetric* systems through active vibration control [20].

All the above-mentioned works dealt with symmetric systems, except [17] that requires knowledge of the system matrices. There are many engineering problems whose stiffness matrices are asymmetric. Usually the asymmetry is produced not by the structure itself alone, but by some external loads interacting with the structure, such as friction in brake noise problems [2–6,21], airflow in aeroelastic flutter problems [22], geometrical coupling in hunting instability of tracked vehicles (trains) [23], follower forces [24], or phase difference between two passes of the cutter in machining causing chatter [25] (also with time delay), or general nonconservative forces on rotors [26].

Ouyang studied assignment of poles (complex eigenvalues) to linear friction-induced vibration through structural modifications using a receptance-based inverse method [27]. By shifting the positive real parts of the complex poles to negative values, an unstable system is stabilised. However, that work showed that it is difficult to assign complex poles to an asymmetric system. Sometimes a solution cannot be found. Sometimes a solution is not physically feasible. Even if a reasonable modification is found, sometimes it may turn out that some other assigned poles become unstable, which is another weakness of passive vibration control.

In this paper, state-feedback control based on receptances of the *symmetric* part of the *asymmetric* systems is used to assign complex poles of asymmetric systems. The real part of the complex poles is of particular interest as it concerns instability of asymmetric dynamics systems. It is an extension to the work by Ram and Mottershead [20] on symmetric dynamic systems. Active mass, active damping, and active stiffness are used. Only linear models are considered so that pole assignment remains a valid approach. It is demonstrated that in principle the unstable vibration that easily occurs in asymmetric systems can be suppressed by the right means of active control (active mass, active damping, or active stiffness) with suitable gains.

2. Pole assignment by state-feedback

The poles of a structure (a symmetric system) with non-negative viscous damping are complex with non-positive real parts. However, an asymmetric system with non-negative viscous damping can have complex poles with positive real parts or even positive real poles, indicating flutter instability or divergence. There have been some works on active control of the level of friction-induced brake noise [28,29], which are a different methodology from the inverse method of assigning complex poles presented in this paper.

The Laplace transform of the second-order asymmetric systems being studied in this paper can be written as

$$(\mathbf{M}s^2+\mathbf{C}s+\mathbf{K}+\sum_{i=1}^{j}\mu_i k_{ci}\mathbf{E}_i)\mathbf{x}(s)=\mathbf{p}(s)+\mathbf{b}u(s) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n\times n$ symmetric mass, damping and stiffness matrices, μ_i, k_{ci} and $\mathbf{E}_i$ are the friction coefficient at the ith degree-of-freedom, and its associated stiffness term and location, and there are j such asymmetric terms; $\mathbf{p}$ and $\mathbf{x}$ are, respectively, the Laplace transforms of the external load vector and displacement vector, and $\mathbf{b}$ is the control force distribution vector. Note that square matrix $\mathbf{E}_i$ has only one non-zero element, whose row corresponds to the tangential degree-of-freedom and whose column corresponds to the normal degree-of-freedom of the same node at the friction interface. For the purpose of demonstrating how to apply the method presented later in the paper, these friction-affected degrees-of-freedom are arranged to be the last j degrees-of-freedom and correspond to the matrix block formed by the last j rows and last j columns of the stiffness matrix.

For state-feedback, the single control input force $u(s)=-(s^2\mathbf{a}^{\mathrm{T}}+s\mathbf{f}^{\mathrm{T}}+\mathbf{g}^{\mathrm{T}})\mathbf{x}$, so that

$$[(\mathbf{M}+\mathbf{b}\mathbf{a}^{\mathrm{T}})s^2+(\mathbf{C}+\mathbf{b}\mathbf{f}^{\mathrm{T}})s+(\mathbf{K}+\mathbf{b}\mathbf{g}^{\mathrm{T}}+\sum_{i=1}^{j}\mu_i k_{ci}\mathbf{E}_i)]\mathbf{x}(s)=\mathbf{p}(s) \tag{2}$$

where $\mathbf{a}$, $\mathbf{f}$, and $\mathbf{g}$ may be called active mass, active damping and active stiffness vectors.

Multiplying both sides of Eq. (2) by the receptance matrix of the symmetric part of the asymmetric system, $\mathbf{H}(s)=(\mathbf{K}+s\mathbf{C}+s^2\mathbf{M})^{-1}$, yields

$$[\mathbf{I}+\mathbf{H}\{\mathbf{b}(\mathbf{a}s^2+\mathbf{f}s+\mathbf{g})^{\mathrm{T}}+\sum_{i=1}^{j}\mu_i k_{ci}\mathbf{E}_i\}]\mathbf{x}(s)=\mathbf{H}\mathbf{p}(s) \tag{3}$$

If the asymmetric stiffness terms are absent, that is, if it is a symmetric system (equivalent to taking μ_i as zero), Eq. (3) may be re-written as

$$\mathbf{x}(s)=\hat{\mathbf{H}}(s)\mathbf{p}(s) \tag{4}$$

where the closed-loop receptance matrix is (extended from Ref. [20])

$$\hat{\mathbf{H}}(s)=\mathbf{H}(s)-\frac{\mathbf{H}(s)\mathbf{b}(\mathbf{g}+s\mathbf{f}+s^2\mathbf{a})^{\mathrm{T}}\mathbf{H}(s)}{1+(\mathbf{g}+s\mathbf{f}+s^2\mathbf{a})^{\mathrm{T}}\mathbf{H}(s)\mathbf{b}} \tag{5}$$

In the derivation of Eq. (5), the Sherman–Morrison formula has been used to take advantage of the property of $\mathbf{b}(\mathbf{g}+s\mathbf{f}+s^2\mathbf{a})^{\mathrm{T}}$ being a rank-one matrix.

For assignment of poles and zeros of a symmetric system, the following equations must be solved, respectively, according to equations below (extended from Ref. [20]):

$$1+(\mathbf{g}+s\mathbf{f}+s^2\mathbf{a})^{\mathrm{T}}\mathbf{H}(s)\mathbf{b}=0 \tag{6}$$

$$\mathbf{H}(s)(1+(\mathbf{g}+s\mathbf{f}+s^2\mathbf{a})^{\mathrm{T}}\mathbf{H}(s)\mathbf{b})-\mathbf{H}(s)\mathbf{b}(\mathbf{g}+s\mathbf{f}+s^2\mathbf{a})^{\mathrm{T}}\mathbf{H}(s)=0 \tag{7}$$

However, when the asymmetric stiffness terms are present in Eq. (3), obviously the Sherman–Morrison formula (or Sherman–Morrison–Woodbury formula) does not provide a neat formula for the closed-loop receptance matrix. Even though Eq. (3) may be written formally as

$$\mathbf{x}(s)=\frac{\mathrm{adj}[\mathbf{I}+\mathbf{H}\{\mathbf{b}(s^2\mathbf{a}+s\mathbf{f}+\mathbf{g})^{\mathrm{T}}+\sum_{i-1}^{j}\mu_i k_{ci}\mathbf{E}_i\}]}{\det[\mathbf{I}+\mathbf{H}\{\mathbf{b}(s^2\mathbf{a}+s\mathbf{f}+\mathbf{g})^{\mathrm{T}}+\sum_{i-1}^{j}\mu_i k_{ci}\mathbf{E}_i\}]}\mathbf{H}\mathbf{p}(s) \tag{8}$$

this formal solution is not helpful as the closed-loop receptance matrix in Eq. (8) involves all n degrees-of-freedom and is very expensive to compute.

A close look at the structure of these asymmetric stiffness terms, however, reveals that they together make up a very low-rank sparse matrix. The rank is not greater than j, which is usually much smaller than n. For example, j=916 while $n\cong 180,000$ in the finite element model of a vented disc brake studied in Ref. [21]. This property was exploited in the structural modifications of asymmetric systems [27] and led to a system of simultaneous linear equations in only a small number of relevant components of $\mathbf{x}(s)$ to be found, whose coefficient matrix is a function of s, μ_i and some elements of $\mathbf{H}(s)$, and also has a low rank. This property allowed assignment of complex poles and critical points of asymmetric systems [27] using passive control (which was partly successful), and will be shown in this paper to allow the assignment of complex poles by active vibration control as well.

To adapt the idea of passive control of Ref. [27] for active control, Eq. (2) is re-arranged as

$$[\mathbf{M}s^2+\mathbf{C}s+\mathbf{K}+\mathbf{b}(s^2\mathbf{a}+s\mathbf{f}+\mathbf{g})^{\mathrm{T}}+\sum_{i=1}^{j}\mu_i k_{ci}\mathbf{E}_i]\mathbf{x}(s)=[\hat{\mathbf{H}}^{-1}+\sum_{i=1}^{j}\mu_i k_{ci}\mathbf{E}_i]\mathbf{x}(s)=\mathbf{p}(s) \tag{9}$$

Multiplying both sides of Eq. (9) by the closed-loop receptance matrix of the *symmetric* system, $\hat{\mathbf{H}}$, yields

$$[\mathbf{I}+\hat{\mathbf{H}}(s)\sum_{i=1}^{j}\mu_i k_{ci}\mathbf{E}_i]\mathbf{x}(s)=\hat{\mathbf{H}}(s)\mathbf{p}(s) \tag{10}$$

Eq. (10) forms the basis of the receptance-based inverse method presented in this paper. The great strength of a receptance-based method is that measured receptances at only a small number of relevant locations are needed [27] and a theoretical (finite element) model, though useful, is not required [12] and hence the thorny issue of the modelling errors can be avoided. Because Eq. (10) does not yield the explicit expression of the closed-loop receptance matrix for asymmetric systems, it is believed to be useful to use specific examples to demonstrate the application of this method, in the next section. The other advantage of this method is the use of some receptance elements of the symmetric system (referred to as statically coupled system in Ref. [6]), which are often measured and available.

3. Application of the receptance-based inverse method

A simulated friction-induced vibration problem (slider–belt) is studied here and shown in Fig. 1 below. It is similar to the model studied by Hoffmann et al. [8].

The system has three masses with m_1 having a degree-of-freedom in the x (horizontal) direction, m_3 having a degree-of-freedom in the y (vertical) direction, and m_2 having degrees-of-freedom in both directions. The belt moves at a constant

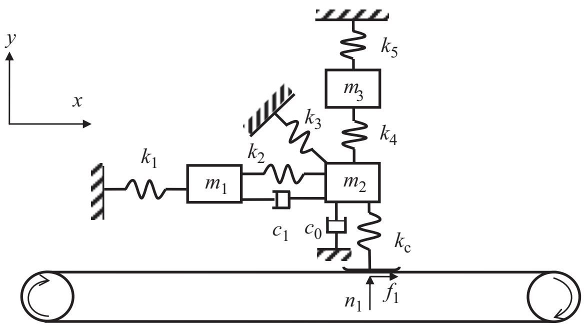

Fig. 1. An asymmetric system of friction-induced vibration.

speed. f_1 and n_1 are, respectively, the friction force and (pre-compression) normal force acting at the slider–belt interface. The sliding friction at the slider–belt interface is governed by Coulomb friction whose static and kinetic friction coefficients are taken to be the same. This is a simplification and avoids stick-slip vibration. **M**, **C**, **K**, and **E** corresponding to displacement vector $x=\{x_1\ y_3\ x_2\ y_2\}^{\mathrm{T}}$ are, respectively:

$$\mathbf{M}=\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_3 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & m_2 \end{bmatrix},\quad \mathbf{C}=\begin{bmatrix} c_1 & 0 & -c_1 & 0\\ 0 & 0 & 0 & 0\\ -c_1 & 0 & c_1 & 0\\ 0 & 0 & 0 & c_0 \end{bmatrix}$$

$$\mathbf{K}=\begin{bmatrix} k_1+k_2 & 0 & -k_2 & 0\\ 0 & k_4+k_5 & 0 & -k_4\\ -k_2 & 0 & k_2+0.5k_3 & -0.5k_3\\ 0 & -k_4 & -0.5k_3 & k_4+0.5k_3+k_c \end{bmatrix},\quad \mathbf{E}=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$$

where $m_i=1\,\mathrm{kg}\ \ (i=1,2,3)$, $c_i=0.5\,\mathrm{N\,s/m}\ (i=0,1)$, $k_i=100\,\mathrm{N/m}(i=1,2,3,4,5)$, and $k_c=2k_1$.

Using the bisection method and MATLAB polyeig function, the critical point of the open-loop system is found to be $\mu_{\mathrm{cr}}=0.3868$, where the system becomes unstable (flutter instability). The proposed method is used below to assign poles to the systems at various μ values.

For this particular example, Eq. (10) becomes

$$\begin{bmatrix} 1 & 0 & 0 & \mu k_c\hat{h}_{13}\\ 0 & 1 & 0 & \mu k_c\hat{h}_{23}\\ 0 & 0 & 1 & \mu k_c\hat{h}_{33}\\ 0 & 0 & 0 & 1+\mu k_c\hat{h}_{43} \end{bmatrix}\begin{Bmatrix} x_1\\ y_3\\ x_2\\ y_2 \end{Bmatrix}=\hat{\mathbf{H}}\mathbf{p} \tag{11}$$

where $\hat{h}_{i3}(i=1,2,3,4)$ are all the elements in the third column of the closed-loop receptance matrix $\hat{\mathbf{H}}$. The complex poles s of the asymmetric system must satisfy the equation below:

$$1+\mu k_c\hat{h}_{43}(s)=0 \tag{12}$$

Substituting Eq. (5) into Eq. (12) and further manipulation of the resultant equation yields

$$[\mathbf{t}^{\mathrm{T}}(s)\ s\mathbf{t}^{\mathrm{T}}(s)\ s^2\mathbf{t}^{\mathrm{T}}(s)]\begin{Bmatrix} \mathbf{g}\\ \mathbf{f}\\ \mathbf{a} \end{Bmatrix}=-1-\mu k_c h_{43}(s) \tag{13}$$

where

$$\mathbf{t}(s)=[1+\mu k_c h_{43}(s)]\mathbf{H}(s)\mathbf{b}-\mu k_c\mathbf{e}_4^{\mathrm{T}}\mathbf{H}(s)\mathbf{b}\mathbf{H}(s)\mathbf{e}_3 \tag{14}$$

and $\mathbf{e}_i$ (i=3, 4) is a vector whose elements are all zero, except its ith element which is one.

Eq. (1) has n pairs of complex poles. If all of them are to be assigned, substitution of them into Eq. (13) leads to $2\times n$ simultaneous equations, which are just enough to solve for n components of any two vectors out of **a**, **f**, and **g**. Better still, additional equations, for example, for assigning zeros and/or modes, or of certain cost functions, may be set up to solve for more than $2\times n$ components of **a**, **f**, and **g**. This paper presents numerical results of only pole assignment with $2\times n$ components of **a**, **f**, and **g**.

3.1. Poles assignment using active damping and active stiffness

The poles to be assigned are $-1\pm 9i$, $-1\pm 13.5i$, $-1\pm 18i$, $-1\pm 22i$ for both the symmetric and asymmetric systems with $\mathbf{b}=\{0\ 0\ 1\ 1\}^{\mathrm{T}}$. Table 1 lists results of **f** and **g** for the symmetric system of $\mu=0$. When they are substituted back into Eq. (2) for $\mu=0$, indeed the poles found are precisely those assigned ones. Actually, assignment of complex poles to symmetric systems using active damping and active stiffness is always successful (please see examples in Ref. [30]).

Table 1
Active damping and stiffness vectors to assign poles $-1\pm 9\mathrm{i}, -1\pm 13.5\mathrm{i}, -1\pm 18\mathrm{i}, -1\pm 22\mathrm{i}$ with $\mathbf{b}$={0 0 1 1}$^{\mathrm{T}}$ at various values of friction coefficients.

μ	Active damping vector **f**	Active stiffness vector **g**
0	$\{1.8\ 6.9\ 3.3\ 3.2\}^{\mathrm{T}}$	$\{-77.0\ 110.7\ 40.9\ 150.4\}^{\mathrm{T}}$
0.3868	$\{2.9\ 5.7\ 5.7\ 0.8\}^{\mathrm{T}}$	$\{-122.8\ 157.5\ 33.8\ 157.0\}^{\mathrm{T}}$
0.45	$\{3.2\ 5.3\ 6.5\ 0.1\}^{\mathrm{T}}$	$\{-135.9\ 171.0\ 31.4\ 159.3\}^{\mathrm{T}}$
0.5	$\{3.4\ 5.0\ 7.2\ -0.7\}^{\mathrm{T}}$	$\{-148.3\ 183.7\ 28.9\ 161.7\}^{\mathrm{T}}$

Table 2
Original and assigned poles at various values of friction coefficient with $\mathbf{b}$={0 0 1 1}$^{\mathrm{T}}$.

Complex poles	First pair	Second pair	Third pair	Forth pair
Original (μ=0)	-0.019 ± 8.028i	-0.047+12.341i	-0.468 ± 16.539i	-0.216 ± 20.233i
Assigned (μ=0)	-1.000 ± 9.000i	-1.000 ± 13.500i	-1.000 ± 18.000i	-1.000 ± 22.000i
Original (μ=0.3868)	0.000 ± 8.733i	-0.053 ± 12.189i	-0.509 ± 16.749i	-0.188 ± 19.857i
Assigned (μ=0.3868)	-1.000 ± 9.000i	-1.000 ± 13.500i	-1.000 ± 18.000i	-1.000 ± 22.000i
Original (μ=0.45)	0.004 ± 8.852i	-0.054 ± 12.159i	-0.518 ± 16.789i	-0.182 ± 19.789i
Assigned (μ=0.45)	-1.000 ± 9.000i	-1.000 ± 13.500i	-1.000 ± 18.000i	-1.000 ± 22.000i
Original (μ=0.5)	0.007 ± 8.946i	-0.055 ± 12.134i	-0.526 ± 16.823i	-0.176 ± 19.734i
Assigned (μ=0.5)	-1.000 ± 9.000i	-1.000 ± 13.500i	-1.000 ± 18.000i	-1.000 ± 22.000i

Table 3
Active damping and stiffness vectors to assign poles $-1\pm9\mathrm{i},-1\pm13.5\mathrm{i},-1\pm18\mathrm{i},-1\pm22\mathrm{i}$ at μ=0.5 with different actuator distributions.

Actuator distribution	Active damping vector **f**	Active stiffness vector **g**
$\mathbf{b}=\{0\ 0\ 0\ 1\}^{\mathrm{T}}$	$\{-17.9\ 9.2\ -17.9\ 6.5\}^{\mathrm{T}}$	$\{262.6\ 37.8\ -243.3\ 192.3\}^{\mathrm{T}}$
$\mathbf{b}=\{1\ 0\ 0\ 0\}^{\mathrm{T}}$	$\{6.5\ -4.7\ 9.0\ -15.9\}^{\mathrm{T}}$	$\{187.8\ -485.3\ -116.1\ 224.2\}^{\mathrm{T}}$
$\mathbf{b}=\{1\ 0\ 1\ 1\}^{\mathrm{T}}$	$\{0.4\ 13.3\ 1.1\ 5.0\}^{\mathrm{T}}$	$\{-11.2\ 577.5\ -245.5\ 448.2\}^{\mathrm{T}}$
$\mathbf{b}=\{1\ 1\ 1\ 1\}^{\mathrm{T}}$	$\{0.0\ 3.4\ 5.5\ -2.5\}^{\mathrm{T}}$	$\{503.1\ -146.5\ -473.9\ 305.1\}^{\mathrm{T}}$

Secondly, assignment of poles of the asymmetric system ($\mu\neq0$) is made. Complex poles of the open-loop asymmetric system at μ_{cr}, $\mu=0.45$ and 0.5 are listed in Table 2 and referred to as ‘original’, referring to the unmodified system. **f** and **g** for assigning the same complex poles of the asymmetric system are given in Table 1. The poles actually obtained (assigned) are given in Table 2.

Table 2 shows that indeed all poles are assigned precisely, even though the real parts have been changed by relatively a large amount. In contrast, this is not possible when using structural modifications studied in Ref. [27], which can make only small changes to the real parts of complex poles of the *asymmetric* system.

Sometimes it is desirable to locate actuators at certain degrees of freedom. Sometimes it is also interesting to know whether one actuator would perform better than more. Different actuator distributions represented by various **b** are simulated. It is found that all these can assign the desired poles precisely. However, the gains required are very different. Table 3 shows the values of active damping and active stiffness for assigning the identical poles to the previous ones at μ=0.5.

If one looks at the maximum values in **g** for different **b** in Tables 1 and 3, obviously three and four actuators are not as good as one actuator. It seems that two actuators require the smallest maximum value in **g**. It is also clear that locating one actuator to different degrees-of-freedom requires different values of active stiffness and active damping. These results indicate that it is possible to determine the optimal number of actuators and optimal actuator locations for assigning complex poles to asymmetric systems using state-feedback control.

3.2. Pole assignment using active damping and active mass

It is found that whatever distributions of actuators, all those complex poles dealt with in Section 3.1 can be assigned precisely, just like using active damping and active stiffness. The results when using two actuators are given in Table 4.

3.3. Pole assignment using active mass and active stiffness

The coefficient matrix thus formed from Eq. (13) for vectors **g** and **a** is ill-conditioned, even for the symmetric system of μ=0. Therefore, it is impossible to assign complex poles using active mass and active stiffness together. This incapability can be explained in a simple example of a mass–spring–damper system as follows:

$$m\ddot{x}+c\dot{x}+kx=0 \tag{15}$$

where m, c, and k are mass, viscous damping, and spring constant, and x is the displacement. The poles of this simple system are known to be

$$s_{1,2}=-\frac{c}{2m}\pm\mathrm{i}\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2} \tag{16}$$

Table 4
Active damping and mass vectors to assign poles with $\mathbf{b}=\{0\ 0\ 1\ 1\}^T$ $-1\pm 9i, -1\pm 13.5i, -1\pm 18i, -1\pm 22i$ at various values of friction coefficients.

μ	Active damping vector **f**	Active mass vector **a**
0	$\{0.99\ 3.20\ 1.39\ 1.32\}^T$	$\{0.09\ -0.43\ -0.19\ -0.35\}^T$
0.3868	$\{1.70\ 3.00\ 2.84\ 0.22\}^T$	$\{0.30\ -0.62\ -0.06\ -0.41\}^T$
0.45	$\{1.90\ 2.87\ 3.30\ -0.19\}^T$	$\{0.36\ -0.68\ -0.02\ -0.44\}^T$
0.5	$\{2.09\ 2.74\ 3.76\ -0.63\}^T$	$\{0.41\ -0.73\ 0.02\ -0.47\}^T$

when the system is under-damped. If the poles to be assigned are $\alpha\pm i\omega$, where α and ω are prescribed numbers, then one gets $\alpha=-c/2m$ and $\omega=\sqrt{k/m-(c/2m)^2}=\sqrt{k/m-\alpha^2}$. This leads to

$$\frac{k}{m}=\omega^2+\alpha^2 \tag{17}$$

Eq. (17) indicates that when a pair of poles is assigned, the ratio of k/m is fixed and there is no way of determining k and m separately. In another word, the equations in k and m will be ill-conditioned or the solutions of mass and stiffness to assign poles will not be unique.

Finally, although a single input force in state-feedback is used in this investigation, it is expected that multiinput forces will precisely assign complex poles too. Output-feedback control has been used to assign poles and zeros to symmetric systems [30]. Its feasibility in assigning poles to asymmetric systems will be studied in near future. Delay can be introduced in both state-feedback and output-feedback [30].

4. Conclusions

This paper studies the assignment of complex poles to friction-induced vibration problems represented by asymmetric second-order dynamic systems, using state feedback control of any two of the three means of active mass, active damping and active stiffness. The inverse method is based on receptances of the symmetric system, which can be directly measured to avoid modelling errors. It is found using a simple simulated example that active damping and active stiffness together is capable of precisely assigning complex poles to asymmetric systems for any distributions of actuators and hence stabilising any unstable poles, and active damping and active mass together is capable of doing the same.

Interestingly, different numbers of actuators at different degrees-of-freedom lead to different values of the active quantities and hence an optimal solution may be obtained. It is also found that active mass and active stiffness together cannot be used to assign complex poles to symmetric or asymmetric systems due to ill-condition of the coefficient matrix of the resultant equations.

It is expected that a multiinput control force of state-feedback, or output-feedback can also precisely assign complex poles to asymmetric systems. The significance of this work is the theoretical demonstration that in principle flutter type vibration that easily occurs in asymmetric systems can be suppressed by suitable mean of active control. The physical tests to demonstrate this capability will be challenging than for symmetric systems as asymmetric systems would normally involve moving components.

Acknowledgement

The author would like to acknowledge the Royal Academy of Engineering/Leverhulme Trust Senior Fellowship for supporting the research reported in this paper.

Dr. Maryam Ghandchi-Tehrani of the Department of Engineering, University of Liverpool, has helped code the MATLAB program used in this investigation. Discussions with her have also been useful.

References

[1] A.F. D'Souza, A.H. Dweib, Self-excited vibration induced by dry friction, part 2: stability and limit-cycle analysis, *Journal of Sound and Vibration* 137 (1990) 177–190.
[2] R.A. Ibrahim, Friction-induced vibration, chatter, squeal, and chaos – part 1: mechanics of friction; part 2: dynamics of modeling, *Applied Mechanics Review* 47 (1994) 209–253.
[3] N.M. Kinkaid, O.M. O'Reilly, P. Papadopoulos, Automotive disc brake squeal: a review, *Journal of Sound and Vibration* 267 (2003) 105–166.
[4] A. Akay, Acoustics of friction, *Journal of the Acoustic Society of America* 111 (2002) 1525–1548.
[5] T. Butlin, J. Woodhouse, Sensitivity of friction-induced vibration in idealised systems, *Journal of Sound and Vibration* 319 (2009) 182–198.
[6] J. Huang, C.M. Krousgrill, A.K. Bajaj, An efficient approach to estimate critical value of friction coefficient and sensitivity analysis for brake squeal, *International Journal of Vehicle Design* 51 (2009) 21–38.
[7] H. Ouyang, L. Baeza, S. Hu, A receptance-based method for predicting latent roots and critical points in friction-induced vibration problems of asymmetric systems, *Journal of Sound and Vibration* 321 (2009) 1058–1068.

[8] N. Hoffmann, M. Fischer, R. Allgaier, L. Gaul, A minimal model for studying properties of the mode-coupling type instability in friction induced oscillations, *Mechanics Research Communications* 29 (2002) 197–205.
[9] N. Hoffmann, L. Gaul, Effects of damping on mode-coupling instability in friction induced oscillations, *ZAMM – Journal of Applied Mathematics and Mechanics* 83 (2003) 524–534.
[10] G. Fritz, J.-J. Sinou, J.-M. Duffal, L. Jézéquel, Investigation of the relationship between damping and mode-coupling patterns in case of brake squeal, *Journal of Sound and Vibration* 307 (2007) 591–609.
[11] U. von Wagner, D. Hochlenert, P. Hagedorn, Minimal models for disk brake squeal, *Journal of Sound and Vibration* 302 (2007) 527–539.
[12] I. Bucher, S. Braun, The structural modification inverse problem: an exact solution, *Mechanical Systems and Signal Processing* 7 (1993) 217–238.
[13] J.E. Mottershead, C. Mares, M.I. Friswell, An inverse method for the assignment of vibration nodes, *Mechanical Systems and Signal Processing* 15 (2001) 87–100.
[14] A. Kyprianou, J.E. Mottershead, H. Ouyang, Structural modification, part 2: assignment of natural frequencies and antiresonances by an added beam, *Journal of Sound and Vibration* 284 (2005) 267–281.
[15] B.N. Datta, S. Elhay, Y.M. Ram, D.R. Sarkissian, Partial eigenstructure assignment for the quadratic pencil, *Journal of Sound and Vibration* 230 (2000) 101–110.
[16] D.J. Inman, Active modal control for smart structures, *Philosophical Transactions of the Royal Society of London A* 359 (2001) 205–219.
[17] E.K. Chu, Pole assignment for second-order systems, *Mechanical Systems and Signal Processing* 16 (2002) 39–59.
[18] L. Benassi, S.J. Elliott, P. Gardonio, Active vibration isolation using an inertial actuator with local force feedback control, *Journal of Sound and Vibration* 276 (2004) 157–179.
[19] J. Qian, S.F. Xu, Robust partial eigenvalue assignment problem for the second-order system, *Journal of Sound and Vibration* 282 (2005) 937–948.
[20] Y.M. Ram, J.E. Mottershead, Receptance method in active vibration control, *American Institute of Aeronautics and Astronautics Journal* 45 (2007) 562–567.
[21] Q. Cao, H. Ouyang, M.I. Friswell, J.E. Mottershead, Linear eigenvalue analysis of the disc-brake squeal problem, *International Journal for Numerical Methods in Engineering* 61 (2004) 1546–1563.
[22] J.R. Wright, J.E. Cooper, *Introduction to Aircraft Aeroelasticity and Loads*, John Wiley & Sons, 2007.
[23] S. Bruni, R. Goodall, T.X. Mei, H. Tsunashima, Control and monitoring for railway vehicle dynamics, *Vehicle System Dynamics* 45 (2007) 743–779.
[24] J.E. Mottershead, S.N. Chan, Flutter instability of circular disc with frictional follower loads, *Journal of Vibration and Acoustics – Transactions of the ASME* 117 (1995) 161–163.
[25] G. Tlusty, *Manufacturing Processes and Equipment*, Prentice-Hall, New Jersey, 2000.
[26] O.N. Kirilov, Campbell diagrams of weakly anisotropic flexible rotors, *Proceedings of the Royal Society of London A* 465 (2009) 2703–2723.
[27] H. Ouyang, Prediction and assignment of latent roots of damped asymmetric systems by structural modifications, *Mechanical Systems and Signal Processing* 23 (2009) 1920–1930.
[28] D. Hochlenert, P. Hagedorn, Control of disc brake squeal-modelling and experiments, *Structural Control and Health Monitoring* 13 (2006) 260–276.
[29] J. Badertscher, K.A. Cunefare, A.A. Ferri, Braking impact of normal dither signals, *Journal of Vibration and Acoustics – Transactions of the ASME* 129 (2007) 17–23.
[30] M. Ghandchi-Tehrani, Passive Modification and Active Vibration Control by the Receptance Method, Ph.D. Thesis, University of Liverpool, 2007.